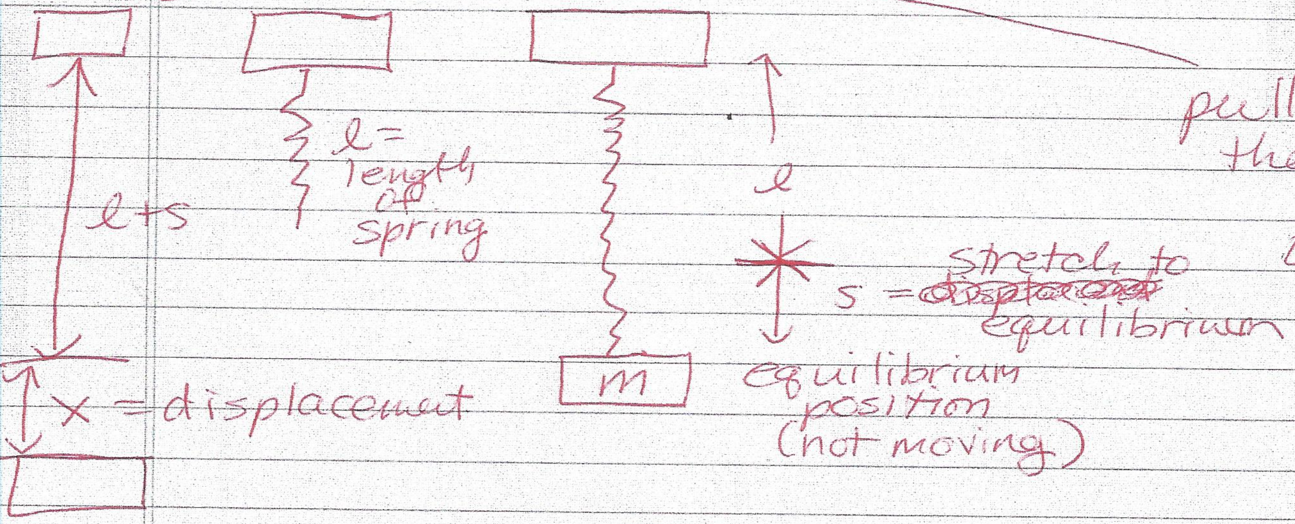
Chapter 5 → project only / not on final or exam.

Section 5.1 Spring/Mass Systems

Free undamped motion



pull on the mass it will oscillate

$$F=ma = \underbrace{m \cdot \frac{d^2x}{dt^2}}_{\text{(mass) (acceleration)}} = -k(s+x) + \underbrace{mg}_{\text{gravitational force}}$$

k — spring constant

$$= -kx + \underbrace{mg - ks}_{0}$$

so $\boxed{m\frac{d^2x}{dt^2} = -kx}$

omega
$\omega^2 = \frac{k}{m}$

Let's relate this to DE

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \quad \text{or} \quad \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

the auxillary eqn is $m^2 + \omega^2 = 0$ $\quad \omega^2 = \frac{k}{m}$
(m from before) $m^2 = -\omega^2$ $\quad$ mass

(be careful with the letter m)

$\boxed{\text{Example 1}}$

Solve the IVP. Find the period and frequency

$$\frac{d^2x}{dt^2} + 49x = 0 \qquad x(0) = 10 \qquad x'(0) = 0$$

spring starts 10 units down at ~~equilibrium~~ (displacement) before it is ~~[illegible]~~ released.

initial velocity

$m^2 + 49 = 0$

$m^2 = -49$

$m = \pm 7i$ and

e^0

$e^0(c_1 \cos 7\cancel{x}^{t} + c_2 \sin 7\cancel{x}^{t})$

$x'(0) = 0$ when released from rest

$x'(0) \neq 0$ if pushed, pulled, etc

$$\boxed{x = c_1 \cos 7t + c_2 \sin 7t} = \boxed{x(t) = 10 \cos 7t}$$

now find c_1, c_2 $\qquad 10 = c_1 \cos 7(0) + c_2 \sin 7(0)$

$$\boxed{c_1 = 10}$$

$x' = -7c_1 \sin 7t + 7c_2 \cos 7t$

$x'(0) = -7c_1 \underbrace{\sin 7(0)}_{0} + 7c_2 \cos 7(0) = 0$

$7c_2 = 0$

$$\boxed{c_2 = 0}$$

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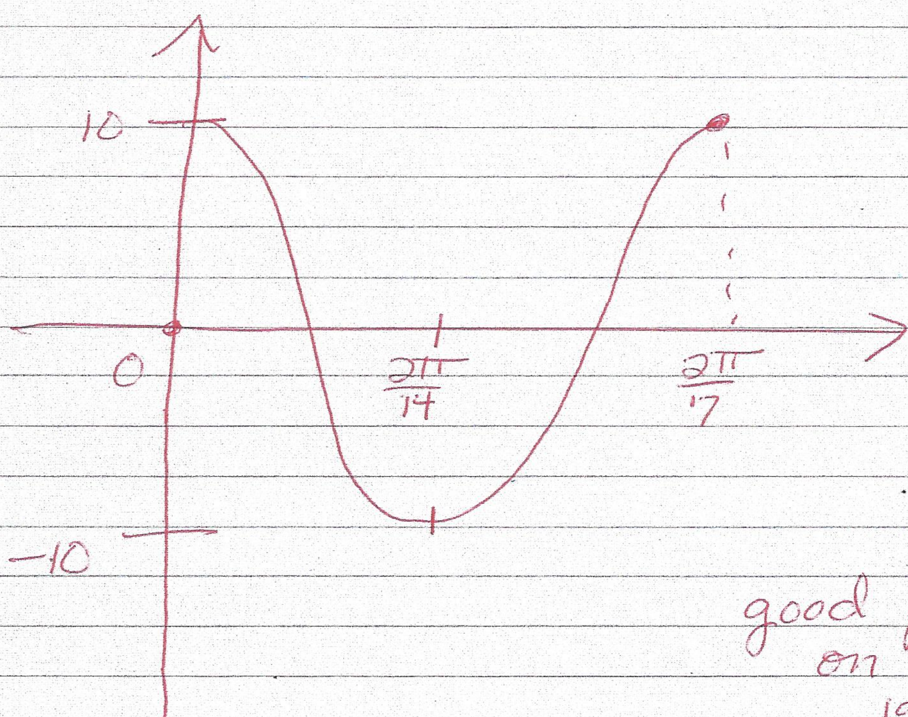
$T = \boxed{\text{period}} = \frac{2\pi}{\omega} = \boxed{\frac{2\pi}{7} \text{ seconds}}$

$\omega^2 = 49$
so $\omega = 7$
(from original problem)

$\boxed{\text{frequency}} = \frac{1}{\text{period}} = \frac{\omega}{2\pi}$

$= \boxed{\frac{7}{2\pi} \text{ cycles/sec} = \text{Hz}}$



good picture on page 196 (undamped motion)

Free Damped motion

damping constant → this force acts in the direction opposite of the motion

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

Let's rewrite this & divide by m

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

m = mass

$\frac{\beta}{m}$: 2λ (Lambda) $\qquad$ $\frac{k}{m}$: ω^2

$\boxed{\frac{\beta}{m} = 2}\lambda = \boxed{2\lambda}$

$\boxed{\frac{k}{m} = \omega^2}$

auxillary eqn: $m^2 + 2\lambda m + \omega^2 = 0$

$a = 1$
$b = 2\lambda$
$c = \omega^2$

solve using quad. eqn.

$D = b^2 - 4ac$

$D = (2\lambda)^2 - 4(1)(\omega^2)$

$= 4\lambda^2 - 4\omega^2$

$= \boxed{4(\lambda^2 - \omega^2) = \Delta}$

$D > 0$ 2 solns (real)
$D = 0$ 1 real (repeats)
$D < 0$ 2 imag. solns.

$$\boxed{m_1} = \frac{-2\lambda - 2\sqrt{\lambda^2 - \omega^2}}{2} = \boxed{-\lambda - \sqrt{\lambda^2 - \omega^2}}$$

$$\boxed{m_2} = \frac{-2\lambda + 2\sqrt{\lambda^2 - \omega^2}}{2} = \boxed{-\lambda + \sqrt{\lambda^2 - \omega^2}}$$

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case 1	$\Delta > 0$	overdamped system (2 real solns)

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$x(t) = c_3 e^{-\lambda t}\left[c_1 e^{(\sqrt{\lambda^2-\omega^2})t} + c_2 e^{(-\sqrt{\lambda^2-\omega^2}}\right.$$

Rewritten

$$x(t) = e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2-\omega^2}\, t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\, t}\right]$$

case 2	$\Delta = 0$	critically damped system (1 repeated soln)

$$x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

$m_1 = m_2$ $\quad x(t) = e^{-\lambda t}(c_1 + c_2 t)$

case 3	$\Delta < 0$	under damped system (2 imag. solns)

$$m_1 = -\lambda + \sqrt{\omega^2-\lambda^2}\; i$$
$$m_2 = -\lambda - \sqrt{\omega^2-\lambda^2}\; i$$

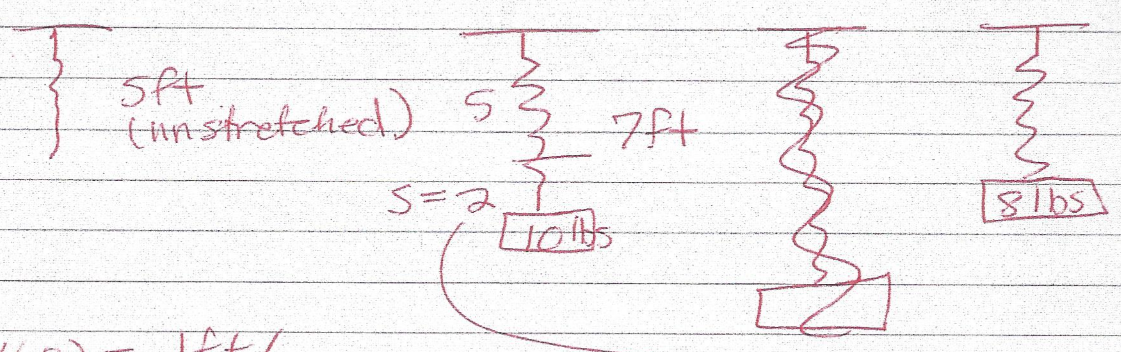
(rearranged)

$$\boxed{x(t) = e^{-\lambda t}\left[c_1 \cos\sqrt{\omega^2-\lambda^2}\; t + c_2 \sin\sqrt{\omega^2-\lambda^2}\; t\right.}$$

We'll look at the graphical situations on Monday

Example

(26) (p. 207) we must build the equation ourselves



$v(0) = 1\,ft/s$

use Hooke's Law

$F = k \cdot s$

$10 = k \cdot (2)$

$5 = k$

mass?

$W = mg$

$(8) = m(32)$

$\frac{1}{4} = m$
(slugs)

our dist. system

Basic form

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

$$\boxed{\frac{1}{4}\frac{d^2x}{dt^2} = -5x - 1\frac{dx}{dt}}$$

β = damping force
= instant. velocity
(from the proble

$\beta = 1$

Initial conditions

$x(0) = \frac{1}{2}$

$x'(0) = 1$